DEFORMS OF LIE ALGEBRAS IN CHARACTERISTIC 2: SEMI-TRIVIAL FOR JURMAN ALGEBRAS, NON-TRIVIAL FOR KAPLANSKY ALGEBRAS

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Abstract. A previously unknown non-linear in roots way to define modulo 2 grading of Lie algebras is described; related are seven new series of simple Lie superalgebras built from Kaplansky algebras of types 2 and 4. This paper helps to sharpen the formulation of the conjecture describing all simple finite dimensional Lie algebras over any algebraically closed field of non-zero characteristic.

We show that certain deformations of simple Lie algebras send some of these algebras into each other; deforms corresponding to non-trivial cohomology classes can be isomorphic to the initial algebra, e.g., proving an implicit Grishkov's claim we explicitly describe Jurman algebra as such semi-trivial deform of the derived of the alternate version of the Hamiltonian Lie algebra, one of the two versions that exist for characteristic 2 together with their divergence-free subalgebras. One of the four types of mysterious Kaplansky algebras is demystified as a non-trivial deform of the alternate Hamiltonian algebra. One more type of Kaplansky algebras is recognized as the derived of another, non-alternate, version of the Hamiltonian Lie algebra, the one that does not preserve any exterior 2-form but preserves a tensorial 2-form.

1. Introduction

1.1. **Main results.** The most interesting part of our paper is the previously unknown *non-linear* in roots way to introduce $\mathbb{Z}/2$-grading in a given simple Lie algebra. It is illustrated with Kaplansky algebras of types 2 and 4; these are the only ones among known Lie algebras succumbing to the new method. Related are seven new series of simple Lie superalgebras.

The main bulk of the paper is devoted to interpretation of the simple Lie algebras discovered by Jurman and Kaplansky in terms of more known examples of Lie algebras of Hamiltonian vector fields. The new notion of semi-trivial deformation is introduced and illustrated with Jurman algebras.

1.2. **Overview.** Hereafter $\mathbb{K}$ is an algebraically closed (unless otherwise specified) field, $\operatorname{char}\mathbb{K} = p$. The stock of non-isomorphic species in the zoo of simple finite dimensional Lie algebras for $p = 2$, was until recently considered uncomfortably numerous (see Introduction to [S]), even despite being incomplete, since it has more exhibits than one would have considered "normal", if we take the case of characteristic $p > 3$, see [BGP], as the "norm".

A conjectural description of all simple finite dimensional Lie algebras over fields of characteristic $p = 2$, recently formulated in detail in [Ltow2], an expounded version of [Ltow], although longer than that for $p > 3$, is possible to grasp; even [Ltow] describes (conjecturally) the classification for $p = 3$. This conjecture gives an overview of the known examples, gathers

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them in describable groups, and indicates the ways to get new examples that definitely are not yet found. A simpler idea, from which this conjecture stemmed, lead to the classification of simple Lie superalgebras of polynomial vector fields over $\mathbb{C}$, see [LSh1].

Several examples (Kaplansky's, Shen's, Jurman's, and Vaughan-Lee's) looked as if they constitute separate items on the list of conjectural ways to construct all simple Lie algebras. This is partly so: the table with the summary of Eick's results on p.948 in a very interesting and important paper [Ei] shows that all new simple algebras of Vaughan-Lee are new only as forms of known Lie algebras over an algebraically closed field (or even over a Galois field larger than $\mathbb{F}_2$); Shen's algebras, see [Sh, Bro], that allow one to work with[1] are, with one remarkable exception[2], called $\mathrm{Bro}_2(1,1)$ in [Ei], either non-simple, or not new, see [LLg].

The paper [Ei] introduced several seemingly new examples that have to be interpreted[3] and described in more detail[4] than they are currently described in [Ei]; perhaps, some of them will be identified with some of the known Lie algebras not taken into account in [Ei] when novelty was being established. The conjecture [Ltow, Ltow2] indicates several directions of search for new Lie algebras and it is absolutely clear that new examples will be unearthed, for example, some of Eick's — tentatively new — algebras might indeed be new (not just forms over non-closed field of some known algebras) though obtainable in a way predicted by the conjecture. But if any of Eick's examples labeled as "new" is of the type not embraced by the conjecture, then this is **very** interesting since the conjecture, irrefutable for three decades by now, should be amended (vagueness and breeziness of the published formulation of the conjecture [Ltow] are among the reasons it was and still is irrefutable).

We started this paper intending to prove that the Jurman and Kaplansky algebras are deforms of more conventional simple Lie algebra such as the two non-isomorphic versions of the Lie algebra of Hamiltonian vector fields, and their divergence-free subalgebras, see [LeP], where they are introduced and interpreted as preserving various types of 2-forms. While this

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[1]The other Shen's algebras do not satisfy the Jacobi identity and we were unable to guess how to amend the typos in the structure constants assuming these typos are reparable.

[2]This exceptional simple Lie algebra Shen introduced is a true analog of the Lie algebra $\mathfrak{g}(2)$ in characteristic 2, whereas the simple-minded reductions of structure constants modulo 2 do not yield a simple Lie algebra or lead to $\mathfrak{psl}(4)$. For clarification of both this statement and Brown's version of the Melikyan's algebras in characteristic 2, see [Shch, GL, BGLLS2].

[3]The standard notation for vectorial Lie algebra with "given name" X realized by vector fields on the m-dimensional space with coordinates $x_1, \dots, x_m$, and shearing vector $\underline{N} = (N_1, \dots, N_m)$ is $X(m; \underline{N})$. In what follows we will often skip the parentheses, e.g., write $\mathfrak{vect}(1; 1)$ instead of $\mathfrak{vect}(1; \underline{(1)})$. We denote the derived of X just by X', not $X^{(1)}$. Usually, except for temporary notation, we stick to Bourbaki's style of notation used in all branches of mathematics and mathematical physics, except the theory of modular Lie algebras, i.e., we use Gothic font for Lie (super)algebras and Latin capitals for Lie groups, e.g., we write $\mathfrak{vect}$ instead of W.

Notation in [Ei] is non-standard and require deciphering: For example, $X(N_1, \dots, N_m)$ should be interpreted as follows: m is the number of indeterminates (say, $x_1, \dots, x_m$) and $(N_1, \dots, N_m)$ are the coordinates of the shearing vector; notation $X(N_1, \dots, N_m)$ is tacitly assumed to refer to an appropriate (first or second) simple derived of the Lie algebra $X(m; (N_1, \dots, N_m))$, rather than to $X(m; (N_1, \dots, N_m))$ itself; for example, $W(2)$ should be read as $W(1; 2)'$ whereas $W(2, 1)$ denotes $W(2; (2, 1))$.

Two arguments in favor of traditional notation: (a) for m large, the redundancy of traditional notation, eliminated as Eick suggests, is helpful to humans although useless for computers and programmers; (b) in many problems, e.g., below in this paper in the description of Kaplansky algebras, to clearly distinguish the algebra from its derived or some other "relative" (such as $\mathfrak{gl}$, $\mathfrak{pgl}$, $\mathfrak{sl}$, $\mathfrak{psl}$, or $\mathfrak{po}$, $\mathfrak{po}'$, $\mathfrak{h}$, $\mathfrak{h}'$) is not just a matter of taste, although in [Ei], where only simple algebras arise, everything is, indeed, clear once deciphered.

[4]Brown's examples were initially described in components only, see [Bro]. In [GL, BGLLS1], they are interpreted together with clarification of their structure and related new simple Lie superalgebras.

paper was being written, Grishkov published a note[5] [GJu] claiming that the Jurman algebra is **isomorphic** to the (derived of) a Hamiltonian Lie algebra. Grishkov's paper is based on a difficult result due to Skryabin, and its main claim on isomorphism is not given explicitly, so we heard doubts if its main result is correct. It is correct: For an explicit isomorphism, see Proposition 3.4 in §3.

Amazingly, the existence of this isomorphism does not contradict the fact that the Jurman algebra is a deform corresponding to a cocycle personifying a non-trivial cohomology class of the (derived of the) Hamiltonian Lie algebra. Here is an example in characteristic 0.

1.3. **On limited information one derives from cohomology in describing deforms of Lie algebras.** In characteristic p, there are known examples of simple Lie algebras $\mathfrak{g}$ such that $H^2(\mathfrak{g};\mathfrak{g}) \neq 0$, all the cocycles constituting a basis of $H^2(\mathfrak{g};\mathfrak{g})$ are integrable, but the deforms are isomorphic to the initial Lie algebra $\mathfrak{g}$, see [BLW]. Moreover, a situation where a non-trivial cocycle describes a linear global deformation, but the deformed algebra is isomorphic to non-deformed one, is possible in any characteristics, including 0. Here is an example independent of characteristics:

Let L be a 2-dimensional Lie algebra with basis $\{e_0, e_1\}$ and the bracket given by $[e_0, e_1] = e_1$. Now consider the Lie algebra $L \otimes (\mathbb{K}[x]/x^2)$. It is a 4-dimensional algebra with basis $\{e_{i,j} \mid i, j = 0, 1\}$, where $e_{i,0} = e_i \otimes 1$ and $e_{i,1} = e_i \otimes x$, and commutation relations

$$
[e_{i,j}, e_{i',j'}] = \begin{cases} (i'-i)e_{i+i',j+j'} & \text{if } i+i', j+j' \leq 1;\\ 0 & \text{otherwise.}\end{cases} \tag{1}
$$

The deform of this algebra given by the expressions

$$
\begin{array}{l} [e_{0,1}, e_{1,1}]_\hbar = \hbar e_{1,0};\\ [e_{i,j}, e_{i',j'}]_\hbar = [e_{i,j}, e_{i',j'}] \text{ for any pair } (e_{i,j}, e_{i',j'}) \text{ distinct from } (e_{0,1}, e_{1,1}) \end{array} \tag{2}
$$

is isomorphic to the initial algebra. Indeed, the following map is an isomorphism between deformed and non-deformed algebras:

$$
\begin{array}{l} M_\hbar(e_{i,0}) = e_{i,0};\\ M_\hbar(e_{i,1}) = e_{i,1} + \sqrt{\hbar}e_{i,0}. \end{array} \tag{3}
$$

The corresponding cocycle $e_{1,0} \otimes d(e_{0,1}) \wedge d(e_{1,1})$ is, however, not a coboundary since in the $\mathbb{Z}$-grading $\deg(e_{i,j}) = j$ this element has weight -2, and there are no 1-cochains of such weight.

We call cocycles with the above property, and the cohomology classes they represent, *semi-trivial*. For more examples of semi-trivial deforms, see §3 on Jurman algebras.

1.4. **Open problems.** 1) The fact that the Jurman algebra is isomorphic to the good old Lie algebra $\mathfrak{h}_\Pi(n; \underline{N})$ of Hamiltonian vector fields, or rather its simple derived,[6] $\mathfrak{h}_\Pi(n; \underline{N})'$, does not make the classification problem of all deforms of $\mathfrak{h}_\Pi(n; \underline{N})'$ meaningless. Investigation of the isomorphism classes of the deforms is a must. The search of the deforms of the more natural non-simple relative of the simple algebra, i.e., of $\mathfrak{h}_\Pi(n; \underline{N})$, not its simple derived, is no less meaningful: it leads to an interpretation of until now mysterious Kaplansky algebras of type 2. Equally reasonable is the search for deforms of another relative, the Poisson Lie

[5] Only now we realized that a draft of this note was available on Grishkov's home page since 2009 or so.

[6] For the description of several types of Lie algebras of Hamiltonian vector fields, their divergence-free subalgebras, the Poisson algebras, and their simple derived in characteristic 2, see [LeP].

Observe a subtlety worth particular attention: the Lie algebra $\mathfrak{po}_\Pi(n; \underline{N})$ is a central extension of $\mathfrak{h}_\Pi(n; \underline{N})$ whereas there is no Lie algebra $\mathfrak{po}_I(n; \underline{N})$, realized on the space of functions, centrally extending $\mathfrak{h}_I(n; \underline{N})$. Indeed, the bracket should be antisymmetric, i.e., alternate, whereas $\{x_i, x_i\}_I = 1$, not 0. For more on possible brackets relative the alternate form B, see in [LeP] and subsec. 2.2.

algebra $\mathfrak{po}_{\Pi}(n;\underline{N})$; these latter deforms are related, in particular, with analogs of spinor representations.

2) The paper [Ei] provides with an approach to the classification[7] allowing a double-check of (sometimes rather sophisticated and therefore difficult to follow and verify) theoretical constructions[8], at least if the structure constants belong to $\mathbb{F}_2$; in other words, the parametric families can not be captured by Eick's method.[9] Although regrettably restricted to algebras of only small dimension (currently 20), Eick's computer-aided approach promises to give — when its range will have been widened to dimension 250, if possible, or at least 80 — a base for the conjectural classification making its theoretical proof psychologically comfortable.

3) It is clear that some of the cocycles describing infinitesimal deformations of $\mathfrak{h}_{\Pi}(n;\underline{N})$ are induced by the quantization of the Poisson algebra, some produce filtered deforms listed by Skryabin [Sk]; these deforms are not isomorphic to the initial algebra and to each other. Are there other cocycles that produce deforms not isomorphic to the initial algebra and the other deforms? Are there such deforms of the simple *derived* of $\mathfrak{h}_{\Pi}(n;\underline{N})$? Our results show that $\mathfrak{h}_{\Pi}(n;\underline{N})$ and $\mathfrak{h}_{\Pi}(n;\underline{N})'$ have different number of deforms and both types of deforms are important for the classification of simple Lie algebras. The situation is similar to that in characteristic 0, where the Lie superalgebra $\mathfrak{h}(2n|m)$ has more deformations than $\mathfrak{po}(2n|m)$, see [LSh2], albeit in one particular superdimension.

4) In [Sk], Skryabin considered only one of the two types of Hamiltonian Lie algebras, $\mathfrak{h}_{\Pi}(n;\underline{N})$. The other type of Hamiltonian Lie algebras, $\mathfrak{h}_{I}(n;\underline{N})$, does not preserve any exterior 2-forms, it preserves a tensorial 2-form, see [LeP]. Investigation à la [Sk] — interpreting filtered deforms of $\mathfrak{h}_{I}(n;\underline{N})$ as vectorial Lie algebras preserving tensorial 2-forms — should be performed for this algebra and its divergence-free subalgebras.

5) Compute the algebraic group $\mathrm{Aut}(\mathfrak{g})$ of automorphisms of $\mathfrak{g}$, thus extending the result of [FG] to the simple Lie algebras without Cartan matrix. This is performed at the moment in certain particular cases only, see Premet's paper [Pre], its continuation in the Ph.D. thesis by M. Guerreiro [GuD], and references therein. To list all non-isomorphic deforms of $\mathfrak{g} = \mathfrak{h}'_{\Pi}(2;g,h+1)$, one has to consider the orbits of $\mathrm{Aut}(\mathfrak{g})$-action on the space $H^2(\mathfrak{g};\mathfrak{g})$ following Kuznetsov and his students, see [KCh, Ch].

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[7]Eick herself does not apply the word "classification" to her method since her search is random and can very well miss something. It is very interesting to estimate the probability of a miss.

[8]Grishkov's paper [GJu] contains also a statement that one of the Kaplansky algebras, the one of dimension 14, provisionally designated as K_{14} by Grishkov and more appropriately as $\mathrm{Kap}_1(4)$ by Eick, is not isomorphic to the simple derived of any of the Hamiltonian Lie algebras preserving the symplectic form well-known from classical mechanics.

One should not think, nevertheless, that the algebra $\mathrm{Kap}_1(4)$ is something undescribable. To describe it, recall that a given symmetric bilinear form B on the space V is said to be *alternate* if $B(v,v)=0$ for any $v\in V$ and *non-alternate* otherwise; the orthogonal Lie algebras that preserve these forms are denoted $\mathfrak{o}_{\Pi}(V)$ and $\mathfrak{o}_{I}(V)$, respectively. The Hamiltonian Lie algebra ($\mathfrak{h}_{\Pi}(V;\underline{N})$ or $\mathfrak{h}_{I}(V;\underline{N})$) is said to be *alternate* or *non-alternate* together with the symmetric bilinear form Π or I preserved by the Lie algebra which is the 0th component in the *standard* $\mathbb{Z}$-grading of the Hamiltonian algebra, the grading in which each indeterminate is of degree 1. These two versions of the Hamiltonian Lie algebras have divergence-free subalgebras described, together with history of earlier partial discoveries, in [LeP]. Instead of $\mathfrak{h}_B(V;\underline{N})$ we write $\mathfrak{h}_B(n;\underline{N})$, where $n=\dim V$.

An isomorphism between $\mathrm{Kap}_1(4)$ and the simple derived of the divergence-free subalgebra $\mathfrak{lh}_{I}(4;\underline{N}_s)$ of $\mathfrak{h}_{I}(4;\underline{N}_s)$, where $\underline{N}_s=(1,\dots,1)$, was indicated already in [Ei].

[9]For example, the Lie algebras like $\mathfrak{wk}(3;a)'/\mathfrak{c}$, where $a\in\mathbb{K}$ and $a\neq 0,1$, see [BGL1], since for $a=0$ the algebra is not simple while for $a=1$ it turns into a simple algebra of different dimension, are invisible to Eick's method and any other method of classification of simple Lie algebras over finite fields.

6) The classification of simple Lie algebras is a problem of interest per se. In particular case of finite dimensional restricted Lie algebras it is related to another, more geometric, problem: classification of simple group schemes, see [Vi1, Vi].

7) There remain several identification problems, see subsections 3.2.2, 3.4.1, 4.2.1, 5.1.1 and 5.3.1.

2. Modular vectorial Lie algebras as deformations of each other

Weisfeiler and Kac discovered first parametric families of simple Lie algebras over $\mathbb{K}$, see [WK]. For further examples of deformations of simple Lie algebras, see [DzhK, Dzh]. In what follows we extend the list of such examples. We will also show that several non-isomorphic Poisson Lie algebras are deforms of one Lie algebra non-simple over $\mathbb{K}$ but simple over a ring, thus resembling forms over algebraically non-closed fields of an algebra defined over an algebraically closed field.

2.1. **Some preparatory information.** Let $\mathbb{K}$ be the ground field of characteristic $p > 0$. In this section we will consider expressions of the form $k \pmod p$ as integers from the segment $[0, p-1]$, not as elements of $\mathbb{K}$.

Lemma. *In the algebra $\mathcal{O}[1;(n)]$, consider a linear map F_α, where $\alpha \in \mathbb{K}$, which acts as follows:*

$$F_\alpha(x^{(k)}) = \alpha^{[\frac{k}{p}]} x^{(k)}, \tag{4}$$

where the square bracket in the expression $[\frac{k}{p}]$ denotes the integer part of $\frac{k}{p}$. If $\alpha \neq 0$, then F_α is an isomorphism of $\mathcal{O}[1;(n)]$ to itself.

Proof. Clearly, F_α is a bijection, so we only need to prove that

$$F_\alpha(x^{(k)} \cdot x^{(l)}) = F_\alpha(x^{(k)}) \cdot F_\alpha(x^{(l)}), \tag{5}$$

i.e.,

$$\alpha^{[\frac{k+l}{p}]} \binom{k+l}{k} x^{(k+l)} = \alpha^{[\frac{k}{p}]+[\frac{l}{p}]} \binom{k+l}{k} x^{(k+l)}. \tag{6}$$

One can see that

$$\begin{array}{ll} \text{if } (k \bmod p) + (l \bmod p) < p, & \text{then } [\frac{k+l}{p}] = [\frac{k}{p}] + [\frac{l}{p}]; \\ \text{if } (k \bmod p) + (l \bmod p) \geq p, & \text{then } \binom{k+l}{k} \equiv 0 \bmod p, \end{array} \tag{7}$$

so in both cases[10] the statement of Lemma holds. □

Consider the endomorphism of $\mathcal{O}[1;(n)]$

$$D_\alpha = F_\alpha^{-1} \circ \partial \circ F_\alpha \text{ explicitly given by the conditions } D_\alpha(x^{(k)}) = \begin{cases} \partial x^{(k)} & \text{if } p \nmid k; \\ \alpha \partial x^{(k)} & \text{if } p \mid k. \end{cases} \tag{8}$$

In what follows, we define D_0 (i.e., D_α for $\alpha = 0$, when F_0 is not defined) using relation (8).

Note that if we consider the isomorphism between $\mathcal{O}[1;(n)]$ and $\mathcal{O}[2;(1, n-1)]$ given by

$$x^{(k)} \longleftrightarrow y_1^{(k \bmod p)} y_2^{([\frac{k}{p}])}, \tag{9}$$

[10] Observe that the thing equal to 0 in the second line of (7) is NOT THE SAME as the thing equal to $[\frac{k}{p}] + [\frac{l}{p}]$ in the first line. Also note that, in the first line, the equality (involving integer parts) is over integers (since the integer parts are used as power degrees); in the second line (involving binomial coefficient), the equality is over $\mathbb{K}$ or modulo p. The inequalities in both lines make sense if the residues of k and l modulo p are considered as integers from the segment $[0, p-1]$.

then D_0 on $\mathcal{O}[1;(n)]$ corresponds to ∂_1 on $\mathcal{O}[2;(1,n-1)]$.

Similarly, in algebra $\mathcal{O}[d;\underline{N}]$ with indeterminates $x=(x_1,\dots,x_d)$, one can consider the map

$$F_\alpha(x^{(r)})=\alpha^{\sum_{i=1}^d[\frac{r_i}{p}]}x^{(r)}, \tag{10}$$

which is an isomorphism for $\alpha\neq 0$; the maps (here $\partial_i:=\partial_{x_i}$)

$$D_{\alpha,i}=F_\alpha^{-1}\circ\partial_i\circ F_\alpha \text{ act as } D_{\alpha,i}(x^{(r)})=\begin{cases}\partial_i x^{(r)} & \text{if } p\nmid r_i;\\ \alpha\partial_i x^{(r)} & \text{if } p\mid r_i.\end{cases} \tag{11}$$

We define $D_{0,i}$ using these relations (11).

2.2. **Poisson Lie algebras.** Consider the Lie algebra $\mathfrak{po}_B(d;\underline{N})$, where $B=(B_{ij})$ is an alternate (anti-symmetric, so to say) non-degenerate bilinear form on a d-dimensional space. The space of this algebra coincides with $\mathcal{O}[d;\underline{N}]$, and the Poisson bracket is defined as

$$[f,g]_B=\sum_{i,j=1}^d B_{ij}\cdot\partial_i f\cdot\partial_j g. \tag{12}$$

Consider the deformed bracket of $\mathfrak{po}_B(d;\underline{N})$ determined by the map F_α on $\mathcal{O}[d;\underline{N}]$ (note that the deformation parameter is $\alpha-1$, not α):

$$\begin{gathered}[f,g]_{B,\alpha}:=F_\alpha^{-1}([F_\alpha(f),F_\alpha(g)])=\sum_{i,j=1}^d B_{ij}F_\alpha^{-1}(\partial_iF_\alpha(f)\cdot\partial_jF_\alpha(g))=\\ \sum_{i,j=1}^d B_{ij}F_\alpha^{-1}(\partial_iF_\alpha(f))\cdot F_\alpha^{-1}(\partial_jF_\alpha(g))=\sum_{i,j=1}^d B_{ij}D_{\alpha,i}f\cdot D_{\alpha,j}g,\end{gathered} \tag{13}$$

since, for $\alpha\neq 0$, the map F_α on $\mathcal{O}[d;\underline{N}]$ preserves the (associative and commutative) multiplication of functions[11].

Now consider the Lie algebra with this bracket (13), the case $\alpha=0$ including. Since we obtained this bracket from a trivial deformation (for $\alpha\neq 0$), the Lie algebra obtained is isomorphic to the initial Lie algebra $\mathfrak{po}_B(d;\underline{N})$. What is the Lie algebra for $\alpha=0$ isomorphic to?

Under the isomorphism between $\mathcal{O}[d;\underline{N}]$ and $\mathcal{O}[2d;(1,\dots,1,N_1-1,\dots,N_d-1)]$ given by

$$x_1^{(r_1)}\dots x_d^{(r_d)}\longleftrightarrow y_1^{(r_1 \bmod p)}\dots y_d^{(r_d \bmod p)}y_{d+1}^{([\frac{r_1}{p}])}\dots y_{2d}^{([\frac{r_d}{p}])}, \tag{14}$$

the operator $D_{0,i}$ on $\mathcal{O}[d;\underline{N}]$ turns into ∂_i on $\mathcal{O}[2d;(1,\dots,1,N_1-1,\dots,N_d-1)]$. So the Lie algebra given by commutation relation (13) with $\alpha=0$ is isomorphic to

$$\begin{gathered}\mathfrak{po}_B(d;(1,\dots,1))\otimes\mathcal{O}[d;(N_1-1,\dots,N_d-1)]\simeq\\ \mathfrak{po}_B(d;(1,\dots,1))\otimes\mathcal{O}[1;(N_1+\dots+N_d-d)]\simeq\\ \mathfrak{po}_B(d;(1,\dots,1))\otimes\mathcal{O}[1;(1)]^{\otimes N_1+\dots+N_d-d}.\end{gathered} \tag{15}$$

[11] The fact that F_α is an isomorphism of $\mathcal{O}[d;\underline{N}]$ to itself (i.e., preserves the multiplication of functions) does NOT imply that F_α produces a deformed bracket: ANY invertible endomorphism of the space $\mathcal{O}[d;\underline{N}]$ produces a deformed bracket. It is the fact that $F_\alpha^{-1}(\partial_iF_\alpha(f)\cdot\partial_jF_\alpha(g))=F_\alpha^{-1}(\partial_iF_\alpha(f))\cdot F_\alpha^{-1}(\partial_jF_\alpha(g))$ that allows the second step in (13).

2.3. **Corollaries.** We see from (15) that all Poisson algebras with the same number of indeterminates, the same $\sum N_i$, and the same bilinear form B (or bilinear forms equivalent over the ground field) are deformations of one Lie algebra.

Conjecturally, the same statement ("a vectorial Lie algebra with a certain $\underline{N}$ is a deformation of the tensor product of its namesake with $\underline{N}_s := (1, \dots, 1)$ and the (associative) function algebra $\mathcal{O}$ with the needed extra $\underline{N}$") is true whenever the space of the Lie algebra is **the same**[12] as that of some $\mathcal{O}$ (or several copies of $\mathcal{O}$), and the bracket can be defined using only derivatives, (associative and commutative) multiplication of functions, and linear operations — e.g., for $\mathfrak{vect}$. The contact bracket also contains multiplications by x_i, but $F_\alpha(x_i) = x_i$, so the same seems to be true for $\mathfrak{k}$ as well.

3. The Jurman algebra is a semi-trivial deform

3.1. **The Jurman algebra.** In [Ju], Jurman introduced a Lie algebra over $\mathbb{F}_2 = \{0, 1\}$ which until now seemed to have no analog over fields $\mathbb{K}$ of characteristic $p \neq 2$. Jurman constructed this algebra by doubling, in a sense, the Zassenhaus algebra, i.e., the derived of the Witt algebra $\mathfrak{vect}(1; \underline{N})$. For this reason he called this algebra *Bi-Zassenhaus algebra* and denoted it $B(g, h)$. But the letter B is overused, besides we wish to emphasize the properties of the Lie algebra $B(g, h)$, different from those Jurman was interested in, so we denote this algebra $\mathfrak{j}(g, h)$ in honor of Jurman. The following description, see [Ju], allows us to extend the ground field and consider $\mathfrak{j}(g, h)$ over $\mathbb{K}$.

Let $g \geq 2$, $h \geq 1$ be integers; $\eta = 2^g - 1$, $k = 2^{g+h} \geq 8$. Considering the elements

$$\{Y_j(t) \mid t \in \{0, 1\},\ \ j \in \{-1, 0, \dots, k-3\}\} \tag{16}$$

as a basis in $\mathfrak{j}(g, h)$ Jurman defined the bracket by setting

$$[Y_i(s), Y_j(t)] = b^{i,j}_{s,t}\, Y_{i+j+st(1-\eta)}(s+t), \tag{17}$$

where (for an elucidation of the meaning of binomial coefficient in the next formula for $s = t = 1$, see Example just below it)

$$b^{i,j}_{s,t} = \begin{cases} \binom{i+j+st(2-\eta)}{i+1} + \binom{i+j+st(2-\eta)}{j+1} & \\ \text{(each binomial coefficient, and their sum,} & \\ \text{are considered modulo 2; meaningless} & \\ \text{expressions should be considered as 0)} & \text{if } -1 \leq i+j+st(2-\eta) \leq k-3, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Example. Consider the case of smallest dimension $(g, h) = (2, 1)$. For $j = -1$, we have the sum $\binom{i-2}{i+1} + \binom{i-2}{0}$. The fist summand has no sense for any i, so should be understood as a 0, the second summand makes sense for $i \geq 2$ when it is equal to 1. For $j = 0$, we have $\binom{i-1}{i+1} + \binom{i-1}{1}$. The first summand makes no sense for any i, the second one makes no sense for $i = -1, 0, 1$; each of these meaningless binomial coefficients should be understood as a 0. If $i > 1$, then $\binom{i-1}{1} \equiv i - 1 \pmod 2$. For $j = 1$, we have $\binom{i}{i+1} + \binom{i}{2}$ with the first summand always meaningless (hence equal to 0) and the second one equal to 0 for $i < 2$.

[12] Not just ISOMORPHIC to $\mathcal{O}$ — for vector spaces it would only mean that they are of the same dimension — but is $\mathcal{O}$ itself, with its extra structures of (associative) multiplication and derivatives.

3.2. **The Jurman algebra $\mathfrak{j}(g,h)$ as a deform of $\mathfrak{h}'_\Pi(2;g,h+1)$.** In order to somehow interpret the Jurman algebra $\mathfrak{j}(g,h)$, we compare it with a known simple Lie algebra; for the most plausible candidates for comparison, see [LeP], where all possible versions of Poisson Lie algebras, and their subquotients — Lie algebras of Hamiltonian vector fields — are described in characteristic 2. We realize the Poisson Lie algebra $\mathfrak{po}_\Pi(2;\underline{N})$ by generating functions (divided powers) in the two indeterminates p and q with the bracket

$$\{f,g\}=\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}-\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}. \tag{19}$$

Consider the Lie algebra of Hamiltonian vector fields $\mathfrak{h}_\Pi(2;\underline{N})=\mathfrak{po}_\Pi(2;\underline{N})/\mathbb{K}\cdot 1$ and its derived $\mathfrak{h}'_\Pi(2;\underline{N})=[\mathfrak{h}_\Pi(2;\underline{N}),\mathfrak{h}_\Pi(2;\underline{N})]$. We keep expressing the elements of $\mathfrak{h}_\Pi$ and $\mathfrak{h}'_\Pi$ by means of generating functions having in mind, by abuse of notation, their classes modulo the center of $\mathfrak{po}_\Pi$.

Recall, see [LSh1], that the *Weisfeiler filtrations* were initially used for description of infinite dimensional Lie (super)algebras $\mathcal{L}$ by selecting a maximal subalgebra $\mathcal{L}_0$ of finite codimension. Dealing with finite dimensional algebras, we can confine ourselves to maximal subalgebras of *least* codimension, or almost least, etc. Let $\mathcal{L}_{-1}$ be a minimal $\mathcal{L}_0$-invariant subspace strictly containing $\mathcal{L}_0$, and $\mathcal{L}_0$-invariant; for $i\geq 1$, set:

$$\mathcal{L}_{-i-1}=[\mathcal{L}_{-1},\mathcal{L}_{-i}]+\mathcal{L}_{-i}\ \text{ and }\ \mathcal{L}_i=\{D\in\mathcal{L}_{i-1}\mid [D,\mathcal{L}_{-1}]\subset\mathcal{L}_{i-1}\}. \tag{20}$$

We thus get a filtration:

$$\mathcal{L}=\mathcal{L}_{-d}\supset\mathcal{L}_{-d+1}\supset\dots\supset\mathcal{L}_0\supset\mathcal{L}_1\supset\dots \tag{21}$$

The d in (21) is called the *depth* of $\mathcal{L}$ and of the associated graded Lie superalgebra $\mathfrak{g}=\bigoplus\limits_{-d\leq i}\mathfrak{g}_i$, where $\mathfrak{g}_i=\mathcal{L}_i/\mathcal{L}_{i+1}$.

Denote $\mathfrak{j}(g,h)$ by $\mathcal{L}$ when considered with a *Weisfeiler filtration*. Eqs. (17), (18) imply that

$$\mathcal{L}_0=\mathrm{Span}(Y_i(0),Y_j(1)\mid i,j\geq 0)$$

is a subalgebra of $\mathcal{L}$; from table (25) we see that $\mathcal{L}_0$ is a maximal subalgebra. The Weisfeiler filtration corresponding to the pair $(\mathcal{L},\mathcal{L}_0)$ is as follows:

$$\mathcal{L}=\mathcal{L}_{-1}\supset\mathcal{L}_0\supset\mathcal{L}_1\supset\mathcal{L}_2\dots,\ \text{ where } \mathcal{L}_{i+1}=\{X\in\mathcal{L}_i\mid [\mathcal{L},X]\subset\mathcal{L}_i\}; \tag{22}$$

let $\mathrm{gr}\,\mathfrak{j}(g,h)=\oplus\,\mathfrak{g}_i$, where $\mathfrak{g}_i=\mathcal{L}_i/\mathcal{L}_{i+1}$ for $i\geq -1$.

3.2.1. Proposition. $\mathrm{gr}\,\mathfrak{j}(g,h)\cong\mathfrak{h}'_\Pi(2;g,h+1)$.

Proof. For any $Y_c(s)\in\mathcal{L}_i$, we designate by $\overline{Y_c(s)}$ its image in the quotient space $\mathfrak{g}_i=\mathcal{L}_i/\mathcal{L}_{i+1}$. Let $0\leq\alpha\leq 2^h-1$ and $0\leq\beta\leq\eta=2^g-1$. Then

$$\overline{Y_i(s)}\longleftrightarrow p^{(\beta)}q^{(2\alpha+1-s)},\ \text{ where } s=0,1;\ i=\alpha(\eta+1)-1-s+\beta. \tag{23}$$

For manual computations, however, it is more convenient to consider the two cases $s=0,1$ separately by setting:

$$\begin{aligned}\overline{Y_a(1)}&\longleftrightarrow p^{(\beta)}q^{(2\alpha)} &&\text{for } a=\alpha(\eta+1)-2+\beta,\\ \overline{Y_b(0)}&\longleftrightarrow p^{(\beta)}q^{(2\alpha+1)} &&\text{for } b=\alpha(\eta+1)-1+\beta.\end{aligned} \tag{24}$$

We see that the maximal power of p is equal to $\eta=2^g-1$ and hence $\underline{N}(p)=g$. Since $2\alpha+1\leq 2^{h+1}-1$, it follows that $\underline{N}(q)=h+1$. □

Accordingly, the basis elements of the components of the first five degrees are as follows:

(25)

$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_3$
$p \longleftrightarrow \overline{Y_{-1}(1)}$	$p^{(2)} \longleftrightarrow \overline{Y_0(1)}$	$p^{(3)} \longleftrightarrow \overline{Y_1(1)}$	$p^{(4)} \longleftrightarrow \overline{Y_{2\eta}(1)}$	$p^{(5)} \longleftrightarrow Y_3(1)$
$q \longleftrightarrow \overline{Y_{-1}(0)}$	$pq \longleftrightarrow \overline{Y_0(0)}$	$p^{(2)}q \longleftrightarrow \overline{Y_1(0)}$	$p^{(3)}q \longleftrightarrow \overline{Y_2(0)}$	$p^{(4)}q \longleftrightarrow Y_3(0)$
	$q^{(2)} \longleftrightarrow \overline{Y_{\eta-1}(1)}$	$pq^{(2)} \longleftrightarrow \overline{Y_\eta(1)}$	$p^{(2)}q^{(2)} \longleftrightarrow \overline{Y_{\eta+1}(1)}$	$p^{(3)}q^{(2)} \longleftrightarrow Y_{\eta+2}(1)$
		$q^{(3)} \longleftrightarrow \overline{Y_\eta(0)}$	$pq^{(3)} \longleftrightarrow \overline{Y_{\eta+1}(0)}$	$p^{(2)}q^{(3)} \longleftrightarrow Y_{\eta+2}(0)$
			$q^{(4)} \longleftrightarrow \overline{Y_{2\eta}(1)}$	$pq^{(4)} \longleftrightarrow Y_{2\eta+1}(1)$
				$q^{(5)} \longleftrightarrow Y_{2\eta+1}(1)$

Denote by $\{\cdot,\cdot\}$ the bracket in $\mathfrak{g} = \mathfrak{h}'_\Pi(2; g, h+1)$ and by $[\cdot,\cdot]$ the bracket in $\mathfrak{j}(g,h)$. Expressing the $\overline{Y_i(s)}$ by means of monomials in p and q we see that, for the simplest case $g = h+1$, the Jurman cocycle is of the form

$$[f,g] = \{f,g\} + c(f,g), \text{ where } c = \sum_{m<n} p^{(\eta)} q^{(m+n-3)} \otimes d(q^{(m)}) \wedge d(q^{(n)}) \in H^2(\mathfrak{g};\mathfrak{g}). \tag{26}$$

The weight of $x \otimes d(y_1) \wedge \dots \wedge d(y_n)$ with respect to a linear combination of the only operator of maximal torus of $\mathfrak{j}(g,h)$, and the operator that $\mathbb{Z}$-grades $\mathfrak{j}(g,h)$, is

$$(\deg_p(x) - 1 - \sum(\deg_p(y_i) - 1),\ \ \deg_q(x) - 1 - \sum(\deg_q(y_i) - 1)). \tag{27}$$

So, the Jurman cocycle is of weight $(2^g, -2)$. By symmetry $p \longleftrightarrow q$, there is a cocycle, of weight $(-2, 2^g)$ leading to an isomorphic Jurman algebra.

If $g \neq h+1$, there is no symmetry $p \longleftrightarrow q$, but there is another Jurman cocycle making $\mathfrak{h}'_\Pi(2; g, h+1)$ into $\mathfrak{j}(h+1, g-1)$. It is of the following form, where $\theta = 2^{h+1} - 1$:

$$c = \sum_{m<n} q^{(\theta)} p^{(m+n-3)} \otimes d(p^{(m)}) \wedge d(p^{(n)}). \tag{28}$$

In characteristic $p > 2$, for simple vectorial Lie algebras $\mathfrak{g}$, most of the cocycles from $H^2(\mathfrak{g};\mathfrak{g})$ are not integrable, see [Dzh]. Here, the situation is completely different:

3.2.2. Lemma (Conjecture for generic values of (g,h)). *Any linear combination of the cocycles from $H^2(\mathfrak{g};\mathfrak{g})$, where $\mathfrak{g} = \mathfrak{h}'_\Pi(2; g, h+1)$, can be integrated to a global deform. Moreover, for $\mathfrak{g} = \mathfrak{h}'_\Pi(2; g, h+1)$, each weight cocycle from $H^2(\mathfrak{g};\mathfrak{g})$ determines a global deform of $\mathfrak{h}'_\Pi(2; g, h+1)$, i.e., each deform corresponding to the weight cocycle is* **linear** *in the parameter of deformation.*

Proof. Computer-aided study for small values of (g,h). $\square$

For $g+h = g'+h' = K$, the Jurman algebras $\mathfrak{j}(g,h)$ and $\mathfrak{j}(g',h')$ considered as $\mathbb{Z}/2$-graded Lie algebras $\mathfrak{j} = \mathfrak{j}_{\bar 0} \oplus \mathfrak{j}_{\bar 1}$ with $\mathfrak{j}_{\bar 0}$ spanned by the $Y_i(0)$ for all i have these even parts isomorphic and the odd parts, as modules over the even part, are also isomorphic; this is clear from eqs. (17), (18). Observe that the brackets of two odd elements given by Jurman's cocycles can be united into one bracket depending on as many parameters as there are partitions $K = g+h$ (with $g \geq 2$, $h \geq 1$); this bracket linearly depends on these parameters. To see this, consider the brackets of two "odd" elements and one "even" element as well as the brackets of three "odd" ones; the statement is obvious in both cases.

3.3. **Deforms of $\mathfrak{h}'_\Pi(2; g, h+1)$ for the smallest values of (g,h).** Obviously, if $g = h+1$, it suffices to consider only cocycles of non-negative weight due to symmetry $p \longleftrightarrow q$.

3.3.1. $(\boldsymbol{g},\boldsymbol{h})=(\mathbf{2},\mathbf{1})$, **i.e., deformations of $\mathfrak{h}'_\Pi(2;2,2)$.** The Jurman cocycle c above is $c_{4,-2}$ from our list (29). The Lie algebra $\mathfrak{h}'_\Pi(2;a,a)$ does have other deforms for sure, e.g., $\mathfrak{psl}(2^a)$ induced by quantization of the Poisson Lie algebra.

Here is a basis of the space $H^2(\mathfrak{g};\mathfrak{g})$, the index of each cocycle is equal to its weight (to save trees, only the cocycles whose expression is short are given in full):

$$
\begin{array}{ll}
c_{4,-2}= & p^{(3)}\otimes d(q)\wedge d(q^{(2)})+p^{(3)}\,q\otimes d(q)\wedge d(q^{(3)})+p^{(3)}\,q^{(2)}\otimes d(q^{(2)})\wedge d(q^{(3)})\\[2mm]
c_{0,-4}= & p\otimes d(p\,q^{(2)})\wedge d(p\,q^{(3)})+p\otimes d(q^{(3)})\wedge d(p^{(2)}\,q^{(2)})+q\otimes d(q^{(3)})\wedge d(p\,q^{(3)}+\\
& p^{(2)}\otimes d(p\,q^{(2)})\wedge d(p^{(2)}\,q^{(3)})+p^{(2)}\otimes d(q^{(3)})\wedge d(p^{(3)}\,q^{(2)})+p\,q\otimes d(q^{(3)})\wedge d(p^{(2)}\,q^{(3)})+\\
& p^{(3)}\otimes d(p^{(2)}\,q^{(2)})\wedge d(p^{(2)}\,q^{(3)})+p^{(3)}\otimes d(p\,q^{(3)})\wedge d(p^{(3)}\,q^{(2)})+p^{(2)}\,q\otimes d(p\,q^{(3)})\wedge d(p^{(2)}\,q^{(3)})\\[2mm]
c_{2,0}= & p^{(2)}\otimes d(p)\wedge d(q)+p\,q^{(2)}\otimes d(q)\wedge d(q^{(2)})+p^{(3)}\,q\otimes d(q)\wedge d(p^{(2)}\,q)+\\
& p^{(3)}\,q^{(2)}\otimes d(p)\wedge d(p\,q^{(3)})+p^{(3)}\,q^{(2)}\otimes d(q^{(2)})\wedge d(p^{(2)}\,q)+p^{(2)}\,q^{(3)}\otimes d(q)\wedge d(p\,q^{(3)})\\[2mm]
c_{0,-2}= & p\otimes d(p)\wedge d(p\,q^{(3)})+p\otimes d(p\,q)\wedge d(p\,q^{(2)})+p\otimes d(q^{(2)})\wedge d(p^{(2)}\,q)+q\otimes d(q)\wedge d(p\,q^{(3)})+\dots\\[2mm]
c_{-2,-2}= & p\otimes d(p\,q^{(2)})\wedge d(p^{(3)}\,q)+q\otimes d(p\,q^{(2)})\wedge d(p^{(2)}\,q^{(2)})+q\otimes d(q^{(3)})\wedge d(p^{(3)}\,q)+\dots
\end{array}\tag{29}
$$

3.3.2. $(\boldsymbol{g},\boldsymbol{h})=(\mathbf{3},\mathbf{1})$ **and** $(\mathbf{2},\mathbf{2})$, **i.e., deformations of $\mathfrak{h}'_\Pi(2;3,2)\simeq\mathfrak{h}'_\Pi(2;2,3)$.** The Jurman cocycle deforming $\mathfrak{h}'_\Pi(2;3,2)$ into $\mathfrak{j}(3,1)$ is $c_{-2,8}$, that deforming $\mathfrak{h}'_\Pi(2;2,3)$ into $\mathfrak{j}(2,2)$ is $c_{4,-2}$ (to save trees, only the cocycles whose expression is short are given in full):

$$
\begin{array}{ll}
c_{0,-8}= & p\otimes\left(d\left(p\,q^{(4)}\right)\wedge d\left(p\,q^{(5)}\right)\right)+p\otimes\left(d\left(q^{(5)}\right)\wedge d\left(p^{(2)}\,q^{(4)}\right)\right)+q\otimes\left(d\left(p\,q^{(4)}\right)\wedge d\left(q^{(6)}\right)\right)+\dots\\
c_{1,-7}= & p\otimes d\left(q^{(4)}\right)\wedge d\left(p\,q^{(4)}\right)+q\otimes d\left(q^{(4)}\right)\wedge d\left(q^{(5)}\right)+p^{(2)}\otimes d\left(q^{(4)}\right)\wedge d\left(p^{(2)}\,q^{(4)}\right)+\dots\\
c_{4,-4}= & p^{(3)}\otimes d\left(q\right)\wedge d\left(q^{(4)}\right)+p^{(3)}\,q\otimes d\left(q\right)\wedge d\left(q^{(5)}\right)+p^{(3)}\,q\otimes d\left(q^{(2)}\right)\wedge d\left(q^{(4)}\right)+\dots\\[2mm]
c_{4,-2}= & p^{(3)}\otimes d\left(q\right)\wedge d\left(q^{(2)}\right)+p^{(3)}\,q\otimes d\left(q\right)\wedge d\left(q^{(3)}\right)+p^{(3)}\,q^{(2)}\otimes d\left(q\right)\wedge d\left(q^{(4)}\right)+\dots\\
c_{1,-5}= & p\otimes d(q^{(2)})\wedge d(p\,q^{(4)})+p\otimes d(p\,q^{(2)})\wedge d(q^{(4)})+\dots\\[2mm]
c_{0,-4}= & p\otimes d(p\,q^{(2)})\wedge d(p\,q^{(3)})+p\otimes d(q^{(3)})\wedge d(p^{(2)}\,q^{(2)})+\dots\\
c_{-1,-5}= & p\otimes d(p^{(2)})\wedge d(p\,q^{(6)})+p\otimes d(p^{(3)})\wedge d(q^{(6)})+\dots\\
c_{-2,-6}= & p\otimes d(p\,q^{(4)})\wedge d(p^{(3)}\,q^{(3)})+q\otimes d(p\,q^{(4)})\wedge d(p^{(2)}\,q^{(4)})+\dots\\[2mm]
c_{-2,-4}= & p\otimes d(p\,q^{(2)})\wedge d(p^{(3)}\,q^{(3)})+p\otimes d(p^{(3)}\,q)\wedge d(p\,q^{(4)})+\dots\\
c_{-1,-3}= & p\otimes d(q^{(2)})\wedge d(p^{(3)}\,q^{(2)})+p\otimes d(p^{(2)}\,q)\wedge d(p\,q^{(3)})+\cdots\\
c_{0,-2}= & p\otimes d(p\,q)\wedge d(p\,q^{(2)})+p\otimes d(q^{(2)})\wedge d(p^{(2)}\,q)+\dots\\
c_{2,0}= & p^{(2)}\otimes d(p)\wedge d(q)+p\,q^{(2)}\otimes d(q)\wedge d(q^{(2)})+\dots\\[2mm]
c_{-2,-2}= & p\otimes d(p\,q^{(2)})\wedge d(p^{(3)}\,q)+q\otimes d(q)\wedge d(p^{(3)}\,q^{(3)})+\dots\\[2mm]
c_{-2,0}= & p\otimes d(p^{(2)})\wedge d(p^{(2)}\,q)+p\otimes d(p\,q)\wedge d(p^{(3)})+\dots\\
c_{-4,-2}= & p\otimes d(p^{(3)})\wedge d(p^{(3)}\,q^{(3)})+q\otimes d(p^{(3)})\wedge d(p^{(2)}\,q^{(4)})+\dots\\[2mm]
c_{-4,0}= & p\otimes d(p^{(3)})\wedge d(p^{(3)}\,q)+q\otimes d(p^{(3)})\wedge d(p^{(2)}\,q^{(2)})+\dots\\
c_{0,4}= & (q^{(4)}\otimes(d(p)\wedge d(q))+(p^{(2)}\,q^{(3)})\otimes d(p)\wedge d(p^{(2)})+\dots\\[2mm]
c_{0,6}= & q^{(6)}\otimes d\left(p\right)\wedge d\left(q\right)+p^{(2)}\,q^{(5)}\otimes d\left(p\right)\wedge d\left(p^{(2)}\right)\\
& +p\,q^{(7)}\otimes d\left(p\right)\wedge d\left(p\,q^{(2)}\right)+p^{(3)}\,q^{(6)}\otimes d\left(p\right)\wedge d\left(p^{(3)}\,q\right)\\
& +p^{(2)}\,q^{(7)}\otimes d\left(q\right)\wedge d\left(p^{(3)}\,q\right)+p^{(2)}\,q^{(7)}\otimes d\left(p^{(2)}\right)\wedge d\left(p\,q^{(2)}\right)\\[2mm]
c_{-2,8}= & q^{(7)}\otimes d(p)\wedge d(p^{(2)})+p\,q^{(7)}\otimes d(p)\wedge d(p^{(3)})+p^{(2)}\,q^{(7)}\otimes d(p^{(2)})\wedge d(p^{(3)})
\end{array}\tag{30}
$$

3.4. Proposition. 1) *The Jurman cocycle is semi-trivial for any g,h. In particular, the cocycle $c_{4,-2}$, see* (29)*, represents the map*

$$(x,y)\mapsto p^{(3)}(\partial_q x\cdot\partial_q^2 y+\partial_q^2 x\cdot\partial_q y),\tag{31}$$

i.e., the deformed bracket is that of the Jurman algebra:

$$[x,y]_\hbar=(\partial_p+\hbar p^{(3)}\partial_q^2)x\cdot\partial_q y+\partial_q x\cdot(\partial_p+\hbar p^{(3)}\partial_q^2)y.\tag{32}$$

2) *The cocycle $c_{0,-4}$, see* (29)*, is semi-trivial.*

3) *The cocycle $c_{2,0}$, see* (29)*, is equivalent to the cochain that represents the map*

$$(x,y)\mapsto p^{(3)}(\partial_q x\cdot\partial_p^2 y+\partial_p^2 x\cdot\partial_q y), \tag{33}$$

(i.e., the difference of these cochains is a coboundary), which is one of deformations obtained by Dzhumadil'daev's method, see [Dzh]*.*

4) *The cocycle $c_{0,-2}$, see* (29)*, is semi-trivial: it is equivalent to the cochain that represents the map*

$$(x,y)\mapsto[\partial_q x,\partial_q y]. \tag{34}$$

5) *The cocycle $c_{-2,-2}$, see* (29)*, is inherited from the quantization of the Poisson Lie algebra. The deformation turns $\mathfrak{h}'_\Pi(2;a,a)$ into $\mathfrak{psl}(2^a)$ for any a.*

Proof. 1) The cocycle is the Jurman one by definition. Let us prove that it is semi-trivial. Here is an explicit isomorphism between $\mathfrak{h}'_\Pi(2;g,h{+}1)$ and $\mathfrak{j}(g,h)$ considered as bi-Zassenhaus algebra. For k and l such that $0\leq k<2^b$ and $0\leq l<2^{a+1}$, set

$$Y_{2^{a+1}k+l-1}(0)=p^{2^a+l}q^k+(k+1)p^lq^{k+1}\longleftrightarrow u^{(2^{a+1}k+l)}\partial_u.$$

The brackets between them are the same as between the basis elements of $\mathfrak{vect}(1;(a+b))$ in "monomial" basis with divided powers of u as coefficients. Accordingly, define the second, "odd" copy of the Zassenhaus algebra, considered as the adjoint module over the first one, by setting

$$Y_{2^{a+1}k+l-1}(1)=p^{2^a+l}q^{k-1}+(k+1)p^lq^k.$$

2) The cocycle $c_{0,-4}$ represents the map

$$(x,y)\mapsto\partial_p\partial_q^2x\cdot\partial_q^3y+\partial_q^3x\cdot\partial_p\partial_q^2y=[\partial_q^2x,\partial_q^2y]. \tag{35}$$

Consider the trivial deformation generated by the series of maps $F_\hbar(x)=x+\hbar Dx$, where $D=\partial_q^2$. Since D is a derivation of $\mathcal{O}[2;2,2]$, D commutes with ∂_p and ∂_q, and $D^2=0$, it follows that the corresponding deformed bracket

$$[x,y]_\hbar^F=F_\hbar^{-1}([F_\hbar(x),F_\hbar(y)])=F_\hbar^{-1}([x,y]+[\hbar Dx,y]+[x,\hbar Dy]+[\hbar Dx,\hbar Dy]) \tag{36}$$

is equal to

$$F_\hbar^{-1}([x,y]+\hbar D([x,y])+\hbar^2[Dx,Dy])=[x,y]+\hbar^2[Dx,Dy]. \tag{37}$$

I.e., the deformed bracket produced by $c_{0,-4}$ is

$$[x,y]_\hbar^{c_{0,-4}}=[x,y]^F_{\sqrt{\hbar}}. \tag{38}$$

This means that the map $F_{\sqrt{\hbar}}$ is an isomorphism between the algebra deformed by $c_{0,-4}$ and the non-deformed algebra.

The catch is, the map $F_{\sqrt{\hbar}}$ is not differentiable with respect to $\hbar$, so the fact that the deform is isomorphic to the initial algebra cannot be observed looking at cohomology.

3) This means that the deformed bracket is equivalent to

$$[x,y]_\hbar=(\partial_p+\hbar p^{(3)}\partial_q^2)x\cdot\partial_q y+\partial_q x\cdot(\partial_p+\hbar p^{(3)}\partial_q^2)y. \tag{39}$$

4) In this case, even though $D^2\neq 0$ for $D=\partial_q$, the derivation D is still nilpotent, so arguments similar to the ones about $c_{0,-4}$ would be applicable to prove that $c_{0,-2}$ is semi-trivial.

5) Obvious. □

3.4.1. **Problem.** Perform interpretation of the non-Jurman cocycles (30), e.g., à la Proposition 3.4.

4. On possible versions of the Jurman algebras

4.1. **Generalizations of the Jurman construction.** Consider a Lie algebra $\mathfrak{a}(2;g,h)$ whose space is $\mathcal{O}[2;(g+h,1)]$, and the bracket is given by the formula (we write x and y in order not to confuse with p and q in previous sections)

$$[u,v]=\partial_x u\cdot(\partial_y+y\partial_x^{2^g})v+(\partial_y+y\partial_x^{2^g})u\cdot\partial_x v=[u,v]_{P.b.}+y(\partial_x u\cdot\partial_x^{2^g}v+\partial_x^{2^g}u\cdot\partial_x v).\tag{40}$$

The Jacobi identity holds since both ∂_x and $\partial_y+y\partial_x^{2^g}$ are a) derivations of $\mathcal{O}[2;(g+h,1)]$ and b) commute with each other (observe in passing that the fact that the conventional Poisson bracket satisfies the Jacobi identity is a corollary of the similar properties of ∂_x and ∂_y).

Take the first derived $\mathfrak{a}'(2;g,h)$ of this Lie algebra, spanned by all monomials except the highest degree element, $x^{(2^{g+h}-1)}y$, and consider the quotient modulo the center $\mathfrak{c}$ generated by constants.

4.1.1. Lemma. *We have $\mathfrak{a}'(2;g,h)/\mathfrak{c}\simeq\mathfrak{j}(g,h)$ with an isomorphism realized by the following expressions:*

$$Y_i(0)=x^{(i+1)}y;\qquad Y_i(1)=x^{(i+2)}.$$

Proof. Direct verification of commutation relations. First, note that the brackets of $Y_i(0)$ with anything do not contain additional terms since these terms do not contain ∂_y but contain multiplication by y, and $Y_i(0)$ already contains y whereas $y\cdot y=0$. Note also that $[Y_i(1),Y_j(1)]_{P.b.}=0$. Taking this into account, we see that

$$\begin{aligned}[Y_i(0),Y_j(0)]=&\ x^{(i)}y\cdot x^{(j+1)}+x^{(i+1)}\cdot x^{(j)}y=\left(\binom{i+j+1}{j+1}+\binom{i+j+1}{i+1}\right)x^{(i+j+1)}y=\\ &\left(\binom{i+j+1}{i+1}+\binom{i+j+1}{j+1}\right)Y_{i+j}(0);\\ [Y_i(0),Y_j(1)]=&\ x^{(i+1)}\cdot xp^{(j+1)}=\binom{i+j+2}{i+1}x^{(i+j+2)},\end{aligned}\tag{41}$$

which is the same as $\left(\binom{i+j+1}{i+1}+\binom{i+j+1}{j+1}\right)Y_{i+j}(1)$ because

if $i+j+1\geq 0$, then $\binom{i+j+1}{i+1}+\binom{i+j+1}{j+1}=\binom{i+j+1}{i+1}+\binom{i+j+1}{i}=\binom{i+j+2}{i+1}$;

if $i+j+1<0$, then $i=j=-1$, and $\binom{i+j+2}{i+1}x^{(i+j+2)}=1$, i.e., is a constant, which generate the center $\mathfrak{c}$, so it is equal to 0 in the quotient $\mathfrak{a}'(2;g,h)/\mathfrak{c}$, and hence, in this case we also have $\left(\binom{i+j+1}{i+1}+\binom{i+j+1}{j+1}\right)Y_{i+j}(1)=0$.

Now, we have

$$\begin{aligned}[Y_i(1),Y_j(1)]=&\ y\left(x^{(i+1)}\cdot x^{(j+1-\eta)}+x^{(i+1-\eta)}\cdot x^{(j+1)}\right)=\\ &\left(\binom{i+j+2-\eta}{i+1}+\binom{i+j+2-\eta}{j+1}\right)x^{i+j+2-\eta}y=\\ &\left(\binom{i+j+2-\eta}{i+1}+\binom{i+j+2-\eta}{j+1}\right)Y_{i+j+1-\eta}(0).\end{aligned}\tag{42}$$

So we see that in all cases the commutation relations are the same as in $\mathfrak{j}(g,h)$. □

4.2. **Comparison with known Lie algebras.** The direct analog of the bracket (40) exists in any characteristic p and looks like this

$$[u,v]=\partial_x u\cdot(\partial_y+y^{p-1}\partial_x^{p^g})v+(\partial_y+y^{p-1}\partial_x^{p^g})u\cdot\partial_x v=[u,v]_{P.b.}+y^{p-1}(\partial_x u\cdot\partial_x^{p^g}v+\partial_x^{p^g}u\cdot\partial_x v).\tag{43}$$

Since for $p>3$, all finite dimensional simple Lie algebras are classified, this bracket should be that of a known Lie algebra.

4.2.1. Problem. *Determine what Lie algebra with the bracket* (43) *is isomorphic to in terms of filtered deformations of the "conventional" Lie algebras of Hamiltonian vector fields, see* [LeP].

4.3. **Generalization that failed.** Observe that $\underline{N}(y)$ may be arbitrary (not equal to 1 as above); accordingly, we can replace $\partial_y + y\partial_x^{2^g}$ by $\partial_y + R(y)\partial_x^{2^g}$, where R is any polynomial (of divided degree not exceeding $\underline{N}(y)$). Conjecturally, only R in the form of the highest possible degree monomial is of interest; the other forms of R can be reduced to this or a constant.

It seems, however, that there is no use to take $\underline{N}(y) > 1$: the result is $\mathfrak{j}(g + N - 1, h)$. Observe that the cocycles that make Jurman algebras from $\mathfrak{h}'_{\Pi}(2; 2, 2)$ and $\mathfrak{h}'_{\Pi}(2; 3, 2)$ change the bracket in precisely this way. A propos, $M(y)$ is of maximal degree there.

4.4. **Generalization that works.** We can consider any number k of pairs of indeterminates with the bracket

$$[u, v] = \sum_{1\leq i\leq k} \partial_{x_i}u \cdot (\partial_{y_i} + y_i\partial_{x_i}^{2^{g_i}})v + (\partial_{y_i} + y_i\partial_{x_i}^{2^{g_i}})u \cdot \partial_{x_i}v. \tag{44}$$

Observe that the g_i can be different for different i.

4.4.1. Lemma. *The Lie algebra $\mathfrak{a}'_{\Pi}(2k; (g_1, h_1), \dots, (g_k, h_k))$ has NO center and NO homogenous ideals for $k = 2$ and $(g_1, h_1) = (g_2, h_2) = (2, 1)$. (Conjecturally, it is simple.)*

Proof. Computer-aided study. □

4.5. $\boldsymbol{\mathfrak{a}_I(2; g, h)}$**.** The Lie algebra $\mathfrak{a}_I(2; g, h)$ based on $\mathfrak{h}_I(2; g + h, 1)$ can also be generalized in the above way by means of the bracket

$$[u, v] = \partial_x u \cdot \partial_x v + (\partial_y + y\partial_x^{2^g})u \cdot (\partial_y + y\partial_x^{2^g})v \tag{45}$$

to begin with and further generalized as indicated above.

4.5.1. Lemma. *The Lie algebra $\mathfrak{a}_I(2k; (g_1, h_1), \dots, (g_k, h_k))$ has NO center and NO homogenous ideals for $k = 2$ and $(g_1, h_1) = (g_2, h_2) = (2, 1)$. (Conjecturally, it is simple.)*

Proof. Computer-aided study. □

The Lie algebra $\mathfrak{a}_I(2; g, h)$ is a deform of $\mathfrak{h}_I(2; g+h, 1)$. To prove this for the smallest values of (g, h), we list all infinitesimal deforms of $\mathfrak{h}_I(2; 2, 2)$. For the cochain $x \otimes (dy_1 \wedge \dots \wedge dy_n)$, its weight is equal to

$$((\deg_p(x) - \sum_{1\leq i\leq n} \deg_p(y_i)) \bmod 2, \quad (\deg_q(x) - \sum_{1\leq i\leq n} \deg_q(y_i)) \bmod 2). \tag{46}$$

Note that this grading comes from a maximal torus, more specifically, from the elements $p^{(2)}$ and $q^{(2)}$. For this reason, it is modulo 2, not over $\mathbb{Z}$. This algebra has also an outer grading $\deg_{out}$ given by

$$\deg(p) = \deg(q) = 1, \quad \deg_{out}(f) = \deg(f) - 2, \quad \deg_{out}(df) = 2 - \deg(f). \tag{47}$$

The cocycles below are all of weight $\{0, 0\}$. They are labeled in accordance with $\deg_{out}$; the superscript numerates cocycles of the same degree, if any such occur.

$$
\begin{aligned}
c^1_{-4} = &\quad p\otimes \big(d(p\,q)\wedge d(p^{(2)}\,q^{(3)})+d(p\,q^{(2)})\wedge d(p^{(2)}\,q^{(2)})+d(p\,q^{(3)})\wedge d(p^{(2)}\,q)\big)+\dots\\
c^2_{-4} = &\quad p\otimes d(p^{(2)}\,q)\wedge d(p^{(3)}\,q)+q\otimes d(p^{(3)})\wedge d(p^{(3)}\,q)+q\otimes d(p^{(2)}\,q)\wedge d(p^{(2)}\,q^{(2)})+\dots\\
c^3_{-4} = &\quad p\otimes d(p\,q)\wedge d(p^{(2)}\,q^{(3)})+p\otimes d(p\,q^{(2)})\wedge d(p^{(2)}\,q^{(2)})+p\otimes d(p\,q^{(3)})\wedge d(p^{(2)}\,q)+\dots\\
c^1_{-2} = &\quad p\otimes d(p^{(2)})\wedge d(p^{(3)})+q\otimes d(p^{(2)})\wedge d(p^{(2)}\,q)+q^{(2)}\otimes d(p^{(2)})\wedge d(p^{(2)}\,q^{(2)})+\dots\\
c^2_{-2} = &\quad p\otimes d(q^{(2)})\wedge d(p\,q^{(2)})+q\otimes d(q^{(2)})\wedge d(q^{(3)})+p^{(2)}\otimes d(q^{(2)})\wedge d(p^{(2)}\,q^{(2)})+\dots\\
c^3_{-2} = &\quad p\otimes d(p^{(2)})\wedge d(p^{(3)})+q\otimes d(p)\wedge d(p^{(3)}\,q)+q\otimes d(p^{(2)})\wedge d(p^{(2)}\,q)+\dots\\
c^4_{-2} = &\quad p\otimes d(p^{(2)})\wedge d(p\,q^{(2)})+p\otimes d(p^{(3)})\wedge d(q^{(2)})+q\otimes d(p^{(2)})\wedge d(q^{(3)})+\dots\\
c_0 = &\quad p\otimes d(q)\wedge d(p\,q)+p^{(2)}\otimes d(q)\wedge d(p^{(2)}\,q))+p^{(3)}\otimes d(q)\wedge d(p^{(3)}\,q)+\dots\\
c^1_2 = &\quad q^{(3)}\otimes d(q)\wedge d(p^{(2)}))+p\,q^{(3)}\otimes d(q)\wedge d(p^{(3)})+p\,q^{(3)}\otimes d(p^{(2)})\wedge d(p\,q)+\dots\\
&\quad +p^{(2)}\,q^{(3)}\otimes d(p^{(2)})\wedge d(p^{(2)}\,q)+p^{(3)}\,q^{(3)}\otimes d(p^{(2)})\wedge d(p^{(3)}\,q)+p^{(3)}\,q^{(3)}\otimes d(p^{(3)})\wedge d(p^{(2)}\,q)\\
c^2_2 = &\quad p^{(3)}\otimes d(p)\wedge d(q^{(2)})+p^{(3)}\,q\otimes d(p)\wedge d(q^{(3)})+p^{(3)}\,q\otimes d(q^{(2)})\wedge d(p\,q)\\
&\quad +p^{(3)}\,q^{(2)}\otimes d(q^{(2)})\wedge d(p\,q^{(2)})+p^{(3)}\,q^{(3)}\otimes d(q^{(2)})\wedge d(p\,q^{(3)})+p^{(3)}\,q^{(3)}\otimes d(q^{(3)})\wedge d(p\,q^{(2)})\\
c^3_2 = &\quad q^{(3)}\otimes d(q)\wedge d(q^{(2)})+p\,q^{(3)}\otimes d(q)\wedge d(p\,q^{(2)})+p\,q^{(3)}\otimes d(q^{(2)})\wedge d(p\,q)\\
&\quad +p^{(2)}\,q^{(3)}\otimes d(q)\wedge d(p^{(2)}\,q^{(2)})+p^{(2)}\,q^{(3)}\otimes d(q^{(2)})\wedge d(p^{(2)}\,q)+p^{(3)}\,q^{(3)}\otimes d(q)\wedge d(p^{(3)}\,q^{(2)})\\
&\quad +p^{(3)}\,q^{(3)}\otimes d(q^{(2)})\wedge d(p^{(3)}\,q))+p^{(3)}\,q^{(3)}\otimes d(p\,q)\wedge d(p^{(2)}\,q^{(2)})+p^{(3)}\,q^{(3)}\otimes d(p\,q^{(2)})\wedge d(p^{(2)}\,q)\\
c^4_2 = &\quad p^{(3)}\otimes d(p)\wedge d(p^{(2)})+p^{(3)}\,q\otimes d(p)\wedge d(p^{(2)}\,q)+p^{(3)}\,q\otimes d(p^{(2)})\wedge d(p\,q)\\
&\quad +p^{(3)}\,q^{(2)}\otimes d(p)\wedge d(p^{(2)}\,q^{(2)})+p^{(3)}\,q^{(2)}\otimes d(p^{(2)})\wedge d(p\,q^{(2)})+p^{(3)}\,q^{(3)}\otimes d(p)\wedge d(p^{(2)}\,q^{(3)})\\
&\quad +p^{(3)}\,q^{(3)}\otimes d(p^{(2)})\wedge d(p\,q^{(3)})+p^{(3)}\,q^{(3)}\otimes (d(p\,q)\wedge d(p^{(2)}\,q^{(2)})+p^{(3)}\,q^{(3)}\otimes d(p\,q^{(2)})\wedge d(p^{(2)}\,q)\\
c_6 = &\quad p^{(3)}\,q^{(3)}\otimes d(p)\wedge d(q)
\end{aligned}
\tag{48}
$$

4.5.1a. Lemma. *For $\mathfrak{g} := \mathfrak{h}_I(2;2,2)$, all cocycles representing the weight elements of $H^2(\mathfrak{g};\mathfrak{g})$, see (48), are integrable, except for c^3_{-2}; moreover, each global deformation determined by the the weight elements of $H^2(\mathfrak{g};\mathfrak{g})$ is linear in the parameter.*

Proof. Computer-aided study. □

5. What Kaplansky algebras are isomorphic to. Non-linear superizations

In 1981, Kaplansky described four (five, actually: the two cases of the fourth type algebras have different dimensions) types of simple Lie algebras for $p = 2$, see [Kap2]. He described them by means of multiplication table only; let us give their interpretation.

Kaplansky defined the algebras in terms of *J-systems* resembling the notion of a root system. Over $\mathbb{F}_2$, define a *J-system* Γ in the space V with a symmetric inner product B as a set of nonzero vectors with the following property: If $u, v \in \Gamma$ are distinct and satisfy $B(u, v) = 1$, then $u + v \in \Gamma$. From any J-system Γ one constructs a Lie algebra $\mathfrak{g}_\Gamma$ over $\mathbb{F}_2$ with basis elements e_u for every $u \in \Gamma$, and the multiplication given by the expressions

$$
[e_u, e_v] = \begin{cases} B(u,v)e_{u+v} & \text{for } u, v \text{ distinct,}\\ 0 & \text{for } u+v \notin \Gamma \text{ or } u = v.\end{cases}
\tag{49}
$$

Each of Kaplansky algebras $\mathrm{Kap}_i(n)$, where $i = 1, 2, 3, 4$, is of the form $\mathfrak{g}_\Gamma$ for some Γ.

Obviously, any algebra defined over $\mathbb{F}_2$ can be defined over $\mathbb{K}$ by extension of the ground field; in what follows, speaking about Kaplansky algebras we assume such extension performed unless otherwise specified.

$\underline{\mathrm{Kap}_1(n)}$: For $n \geq 4$, let $\dim V = n$ and assume that V carries a non-degenerate and non-alternate inner product B. Let $e_l, \dots, e_n$ be an orthonormal basis of V. For Γ take all

vectors in V except 0 and $e_l+\dots+e_n$ (A basis-free description of the element to be omitted is as follows: the unique x satisfying $B(x,y)=B(y,y)$ for all y.)

Recall, see [LeP], that $\mathfrak{lh}_I(n;\underline{N})$ is the Cartan prolong of $\mathfrak{o}_I(n)'$. As is easy to see, $\mathrm{Kap}_1(n)=\mathfrak{lh}_I(n;\underline{N}_s)'$, where $\underline{N}_s=(1,\dots,1)$ and where instead of monomials in x_i Kaplansky considered monomials in $X_i:=1+x_i$. (In particular, this yields an interpretation of $K_{14}=\mathrm{Kap}_1(4)$ sought for but not found in [Ju, GJu]; an isomorphism $\mathrm{Kap}_1(n)=\mathfrak{lh}_I(n;\underline{N}_s)'$ was established in [Ei], although in different terms.)

$\underline{\mathrm{Kap}_2(2m)}$: Let $\dim V=2m$ and assume that V carries a non-degenerate and alternate inner product Π, and take all nonzero vectors in V. Kaplansky mentioned this algebra because it fits into the approach he suggested although this algebra has analogs for any characteristic[13] $p>0$, so we could ignore it; it is a filtered deform of $\mathfrak{h}_\Pi(2m;\underline{N}_s)$. If we had ignored it, we would be unable to discover a new notion of non-linear superization.

$\underline{\mathrm{Kap}_3(n)=\mathfrak{o}_I(n)'}$, where $n=5$, 7, and ≥ 9 as Kaplansky observed himself (in different terms). Kaplansky wrote "the gaps avoid duplication", this accounts for omission of $n=3$ as well.

$\underline{\mathrm{Kap}_{4,A}(2m)\text{, where } A=0 \text{ or } 1}$, is a temporary, for the lack of better idea, notation of the two similarly described but equally mysterious algebras:

Let $\dim V=2m$, where $m\geq 3$, and Q a non-degenerate quadratic form on V. We form a J-system $\Gamma=\{u\in V\mid Q(u)=1\}$ relative to the alternate bilinear form B attached to Q:

$$B(u,v)=Q(u+v)+Q(u)+Q(v). \tag{50}$$

Set (for the most lucid definition of Arf invariant, see [Dye] in eq. (51), A is the value of Arf invariant whereas B is just a label, short for "Big")

$$\begin{aligned}&\Gamma_A=\{u\in V\mid Q(u)=1\},\ \text{where } Q \text{ is such that } \mathrm{Arf}(Q)=A;\\ &\mathrm{Kap}_{4,A}(2m):=\mathfrak{g}_{\Gamma_A}(2m),\\ &\mathrm{Kap}_{4,B}(2m):=\mathfrak{g}_{\Gamma}(2m).\end{aligned} \tag{51}$$

5.1. Proposition. 1) *The Lie algebra* $\mathrm{Kap}_{4,B}(2m)$ *is isomorphic to the algebra whose space is* $\mathcal{O}[2m;\underline{N}_s]$ *with indeterminates* p_i,q_i, *where* $1\leq i\leq m$, *and the bracket*

$$[f,g]=\sum_{1\leq i\leq m}(1+p_i)(1+q_i)(\partial_{p_i}f\cdot\partial_{q_i}g+\partial_{q_i}f\cdot\partial_{p_i}g). \tag{52}$$

2) *The Lie algebra* $\mathrm{Kap}_{4,B}(2m)$ *is also isomorphic to a deformed Poisson algebra* $\mathfrak{po}_\Pi(2m;\underline{N}_s)$ *with the deformed bracket*

$$[f,g]_\hbar=\sum_{1\leq i\leq m}(1+\hbar p_iq_i)(\partial_{p_i}f\cdot\partial_{q_i}g+\partial_{q_i}f\cdot\partial_{p_i}g)\ \textit{for any}\ \hbar\neq 0 \tag{53}$$

and

$$\mathrm{Kap}_{4,B}(2m)/\mathfrak{c}\simeq\mathrm{Kap}_2(2m). \tag{54}$$

Proof. 1) The isomorphism is given as follows: Choose a symplectic basis for the inner product B in V. If $(u_1,\dots,u_{2m})$ are coordinates of a vector $u\in V$ in this basis, then

$$e_u\longleftrightarrow f_u=(1+p_1)^{u_1}\dots(1+p_m)^{u_m}(1+q_1)^{u_{m+1}}\dots(1+q_m)^{u_{2m}}.$$

[13] Kaplansky did not give any explicit description of such algebras. Here it is for any $p>0$: Consider the polynomial algebra in $y_i:=\exp(x_i)$, where we set $\partial_{x_i}y_j=\delta_{ij}y_j$ and $(y_i)^p=\exp(px_i)=1$. In the space of these exponentials, introduce the usual Poisson bracket and consider the quotient modulo constants.

2) Clearly, (52) is a particular case of the bracket

$$[f,g]_\hbar = \sum_{1\leq i\leq m} (1+\hbar' p_i)(1+\hbar' q_i)(\partial_{p_i} f\cdot\partial_{q_i} g + \partial_{q_i} f\cdot\partial_{p_i} g) \text{ with } \hbar' = 1. \tag{55}$$

Here, the part linear in $\hbar'$ describes a trivial (as can be checked) deformation of $\mathfrak{po}_\Pi(2m;\underline{N}_s)$, and the quadratic part corresponds to (53) with $\hbar = (\hbar')^2$; this cocycle is non-trivial as a computer-aided study shows. □

5.1.1. **How to establish non-isomorphicy?.** Skryabin [Sk] classified the filtered deforms of Hamiltonian Lie algebras $\mathfrak{h}_\Pi(2m;\underline{N})$; it remains to select which of them is the simple Lie algebra $\mathrm{Kap}_{4,B}(2m)/\mathfrak{c}\simeq\mathrm{Kap}_2(2m)$. We did not perform such an identification yet.

Given two Lie algebras of the same dimension, to find out if they are isomorphic, Eick considered the following invariants in [Ei]:[14] $\dim H^1(\mathfrak{g};\mathfrak{g})$ or rather $\dim\mathfrak{der}(\mathfrak{g})$, the order of the group $\mathrm{Aut}(\mathfrak{g})$, the number of elements in $\mathrm{Ann}(\mathfrak{g})$ and the order of $\mathrm{Exp}(\mathfrak{g})$.

Speaking of deforms, one can consider the action of $\mathrm{Aut}(\mathfrak{g})$ on the space of infinitesimal deformations, as in [KCh, Ch].

At least theoretically, for algebras of small dimension, there is still another approach: Compare identities the algebras satisfy. A. A. Kirillov formulated the following analog of the Amitsur-Levitzki theorem, proof of which was only preprinted in Keldysh Inst. of Applied Math. in 1980s; for a translation of one such preprint, see [KOU]; the other preprints with related results by Kirillov, Kontsevich and Molev had not been translated yet but at least are reviewed by Molev. Dzhumadildaev suggested an interesting modification of emphasis in this train of thought finding a hidden supersymmetry and a relation to strongly homotopy algebras; for details, see [LL].

Theorem ([Ki]). *Let $\mathfrak{g}$ be a simple Lie algebra of vector fields over a field of characteristic 0. Let*

$$a_k(x_1,\dots,x_k) = \sum_{\sigma\in S_k} (-1)^{\operatorname{sign}\sigma} x_{\sigma(1)}\dots x_{\sigma(k)}. \tag{56}$$

The identity $a_k(x_1,\dots,x_k)\equiv 0$ for any $x_1,\dots,x_k\in\mathfrak{g}$ holds
a) for $k\geq (n+1)^2$ if $\mathfrak{g}=\mathfrak{vect}(n)$,
b) for $k\geq n(2n+5)$ if $\mathfrak{g}=\mathfrak{h}(2n)$,
c) for $k\geq 2n^2+5n+5$ if $\mathfrak{g}=\mathfrak{k}(2n+1)$.

5.1.1a. Conjecture. *The Lie algebra $\mathrm{Kap}_{4,B}(2m)$ is not isomorphic to $\mathfrak{po}_\Pi(2m;\underline{N}_s)$ and $\mathrm{Kap}_2(2m)$ is not isomorphic to $\mathfrak{h}_\Pi(2m;\underline{N})$.*

We checked this for m small: For $m=1$, $\mathrm{Kap}_{4,B}(2)$ is isomorphic to $\mathfrak{o}'(3)\oplus\mathfrak{c}$, where $\mathfrak{c}$ is the 1-dimensional trivial center, and thus not isomorphic to $\mathfrak{po}_\Pi(2;\underline{N}_s)$ which is solvable; for $m=2$, a computer-assisted computations show that the infinitesimal deformation corresponding to (53) is a non-trivial cocycle. To prove the conjecture, we have to show that the cocycle is not semi-trivial, either. Of course, what we really need to know is what $\mathrm{Kap}_{4,B}(2m)$ and its subalgebras $\mathrm{Kap}_{4,A}(2m)$ ARE isomorphic to.

[14] Almost quotation from [Ei]: "We say that derivation $d\in\mathfrak{der}(\mathfrak{g})$ is p-nilpotent if $d^p=0$ holds. For a p-nilpotent derivation d, we define its exponential matrix $\exp d := \sum\limits_{0\leq i\leq p-1}\frac{d^i}{i!}$. We call a p-nilpotent derivation d an *annihilator* if $d^i(x)d^j(y)=0$ for all $x,y\in\mathfrak{g}$ and $i,j\geq 0$ with $i+j\geq p$. Let $\mathrm{Ann}(\mathfrak{g})\subset\mathfrak{der}(\mathfrak{g})$ denote the subset of annihilators. We define $\mathrm{Exp}(\mathfrak{g})$ as the subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by $\{\exp(d)\mid d\in\mathrm{Ann}(\mathfrak{g})\}$. Note that the order of every element $\exp(d)$ is equal to either p or 1. Thus $\mathrm{Exp}(\mathfrak{g})$ is a subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by automorphisms of order p."

5.1.1b. Conjecture. *The Lie algebra* $\mathrm{Kap}_{4,1}(2m)$ *is a deform of the subalgebra in* $\mathfrak{po}(2m;\underline{N}_s)$ *consisting of functions* f *satisfying* $\sum\limits_{1\leq i\leq 3}\frac{\partial^2 f}{\partial p_i\partial q_i}=0$. *(The quotient of this subalgebra modulo center is isomorphic to* $\mathfrak{slh}(2m)$*, see* [LeP]*.)*

The dimension of the space of infinitesimal deformations $H^2(\mathfrak{g};\mathfrak{g})$ is big and grows quickly with m. How to select the needed deform? The Poisson algebra, and its subalgebra consisting of harmonic functions, have center generated by constants, whereas $\mathrm{Kap}_{4,1}(2m)$ is simple. Therefore, in the huge space of cocycles representing infinitesimal deformations, we only have to select the cocycles of the form

$$f\otimes d(1)\wedge dg+\dots\ , \tag{57}$$

and compare the global deforms corresponding to such cocycles with $\mathrm{Kap}_{4,1}(2m)$. The dimension of the space $H^2(\mathfrak{g};\mathfrak{g})$ does not yet explode for m small: For $m=2$ and $m=3$, we have $\dim H^2(\mathfrak{g};\mathfrak{g})=34$; all cocycles are integrable and all global deforms corresponding to them (if a representative is chosen carefully by means of coboundaries) are linear in parameter of deformation. For $m=2$ and 3, there is only one (up to coboundaries) cocycle of the form (57). These cocycles are of degree 2. In degree 2, there is only one cocycle for $m=3$ and 5 cocycles for $m=2$. Further investigations show that the conjecture is true for $m=2$ but for $m=3$ the two algebras to be compared have different number of central extensions. Therefore, in the final version of this paper this conjecture will be excluded. Tempting is the coincidence of dimensions of the following algebras.

5.1.1c. Conjecture. *The Lie algebra* $\mathrm{Kap}_{4,1}(2m)$ *is a deform of* $\mathfrak{o}'_I(2m+1;\underline{N}_s)$ *whereas* $\mathrm{Kap}_{4,0}(2m)$ *is a deform of the subalgebra in* $\mathfrak{o}'_I(2m;\underline{N}_s)$*, see* [LeP].

5.1.2. **Kaplansky algebras $\mathrm{Kap}_{4,B}(2m)$ and $\mathrm{Kap}_{4,A}(2m)$ in convenient indeterminates.** Here are examples of the forms Q_A with Arf invariant equal to A:

$$\begin{aligned}Q_0(u)&=\sum_{1\leq i\leq m}u_iu_{m+i},\\ Q_1(u)&=u_1^2+u_{m+1}^2+\sum_{1\leq i\leq m}u_iu_{m+i},\end{aligned}\tag{58}$$

The subalgebras $\mathrm{Kap}_{4,A}(2m)\subset\mathrm{Kap}_{4,B}(2m)$ with bracket (52) are spanned by the non-zero elements f_u such that $Q_A(u)=1$; from the definition (51) we derive the following conditions that single out the subalgebras $\mathrm{Kap}_{4,A}(2m)$ in $\mathrm{Kap}_{4,B}(2m)$:

$$\begin{aligned}&f+\sum_{1\leq i\leq m}(1+p_i)(1+q_i)\partial_{p_i}\partial_{q_i}f=0 &&\text{for }\mathrm{Kap}_{4,0}(2m);\\ &f+(1+p_1)\partial_{p_1}f+(1+q_1)\partial_{q_1}f+\sum_{1\leq i\leq m}(1+p_i)(1+q_i)\partial_{p_i}\partial_{q_i}f=0 &&\text{for }\mathrm{Kap}_{4,1}(2m).\end{aligned}\tag{59}$$

Kaplansky claimed (this follows from (58)) that

$$\dim\mathfrak{g}_{\Gamma_A}=2^{m-1}(2^m-(-1)^A)=\begin{cases}2^{m-1}(2^m-1) & \text{if } \mathrm{Arf}(Q)=0,\\ 2^{m-1}(2^m+1) & \text{if } \mathrm{Arf}(Q)=1.\end{cases}\tag{60}$$

Now let us study the structure of these algebras. It is more convenient to pass to coordinates $x_i:=(1+p_i)$ and $y_i:=(1+q_i)$. The bracket (52) and operators (59) become

$$[f,g]=\sum_{1\leq i\leq m}x_iy_i(\partial_{x_i}f\cdot\partial_{y_i}g+\partial_{y_i}f\cdot\partial_{x_i}g).\tag{61}$$

and

$$(62)\qquad \begin{array}{ll} f+\sum\limits_{1\le i\le m} x_iy_i\partial_{x_i}\partial_{y_i}f=0 & \text{for } \mathrm{Kap}_{4,0}(2m);\\ f+x_1\partial_{x_1}f+y_1\partial_{y_1}f+\sum\limits_{1\le i\le m} x_iy_i\partial_{x_i}\partial_{y_i}f=0 & \text{for } \mathrm{Kap}_{4,1}(2m). \end{array}$$

For example,

(63)

$$\begin{array}{ll} \mathrm{Kap}_{4,0}(2)=\mathrm{Span}(x_1y_1), & \mathrm{Kap}_{4,1}(2)\simeq\mathfrak{o}'_\Pi(3)\simeq\mathfrak{vect}'(1;(2))=\mathrm{Span}(x_1,\ y_1,\ x_1y_1);\\ \mathrm{Kap}_{4,0}(4)\simeq\mathfrak{o}'_\Pi(3)\oplus\mathfrak{o}'_\Pi(3), & \mathrm{Kap}_{4,1}(4)\simeq\mathrm{Kap}_3(5)=\mathfrak{o}'_\Pi(5). \end{array}$$

5.1.2a. Remark. Clearly, the Kaplansky algebras $\mathrm{Kap}_{4,A}(2m)$ are $(\mathbb{Z}/2)^{2m}$-graded by degrees modulo 2 with respect to each indeterminate x_i and y_i.

5.1.3. **The invariant symmetric bilinear forms.** Kaplansky also claimed that these Lie algebras possess a non-degenerate invariant bilinear symmetric form — let us designate it by K — and several other interesting properties verification of which “is quite routine”. Unlike Kaplansky, we think that a lucid proof of these properties is also of interest; here we prove the existence of the invariant form K. The description of K in presence of the alternate form B — i.e., for Kaplansky algebras of types 2, 3 and 4 — is very simple:

$$(64)\qquad K(e_u,e_v)=\delta_{u,v}.$$

The form K is invariant, i.e.,

$$K([e_u,e_z],e_v)=K(e_u,[e_z,e_v])$$

because

$$\begin{array}{ll} \text{if } u+z\neq v, & \text{then } u\neq z+v, \text{ and both sides vanish;}\\ \text{if } u+z=v \text{ (and } u=z+v), & \text{then the l.h.s. is } K(B(u,z)e_v,e_v)=B(u,z),\\ & \text{the r.h.s. is } B(z,v)=B(z,u+z)=B(z,u)\\ & \text{since the form } B \text{ is alternate, and hence } B(z,z)=0. \end{array}$$

We can not guess how Kaplansky reasoned for the case of non-alternate form B. Our argument relies on the invariant form on the Poisson Lie algebra induced by (the “desuperization” of) the Berezin integral[15]

$$(65)\qquad K(f,g)=\int fg:=\text{the coefficient of the highest term,}$$

if the Poisson algebra $\mathfrak{po}_I(n;\underline{N}_s)$ is considered as a “desuperization” of the Lie superalgebra $\mathfrak{po}(0|n)$, i.e., if the space of $\mathfrak{po}(0|n)$, the Grassmann superalgebra, is identified with the algebra of truncated polynomials in even indeterminates.

5.2. **Restricted closure of Kaplansky algebras.** Over $\mathbb{F}_2$, the 2-closure of $\mathrm{Kap}_2(2m)$ and $\mathrm{Kap}_{4,A}(2m)$, except $\mathrm{Kap}_{4,0}(2)$, can be described as follows: Let the space of the closure be $\mathfrak{g}\oplus V^*$, where $\mathfrak{g}$ is the algebra to be closed, and set:

$$(66)\qquad [\alpha,\beta]=0;\qquad [\alpha,e_u]=\alpha(u)e_u \text{ for any } \alpha,\beta\in V^*,\ e_u\in\mathfrak{g}.$$

For a fixed $u\in V$, let $B_u\in V^*$ be the map

$$(67)\qquad B_u: v\mapsto B(u,v) \text{ for any } v\in V.$$

[15]For a short summary of basics on Linear Algebra and Geometry in super setting, see [LSh1]; for a textbook, see [Lsos].

Then we can define squaring by setting

$$\alpha^{[2]} = \alpha; \qquad e_u^{[2]} = B_u \in V^*. \tag{68}$$

Over an arbitrary field $\mathbb{K}$ of characteristic 2 the space of the closure is also $\mathfrak{g} \oplus V^*$, but $\mathfrak{g}$ and V^* are considered over $\mathbb{K}$, and squaring is given by the formula:

$$(a\alpha)^{[2]} = a^2\alpha; \qquad (ae_u)^{[2]} = a^2 B_u \in V^* \text{ for any } a \in \mathbb{K}. \tag{69}$$

This description of the restricted closure clearly shows that none of the Lie algebras $\mathrm{Kap}_{4,A}(2m)$ for $m > 2$ is isomorphic to the simple derived of the orthogonal Lie algebra of the same dimension. Indeed, the restricted closures of these algebras are of different dimensions: The co-dimension of the simple derived of the orthogonal algebra in its restricted closure is much greater than $\dim V^*$.

5.3. **General remark on superizations of restricted Lie algebras.** In characteristic 2 ANY $\mathbb{Z}/2$-grading turns any restricted Lie algebra into a Lie superalgebra.

The only known (until this paper) way to obtain a $\mathbb{Z}/2$-grading on a Lie algebra boils down to the following: Take an arbitrary linear function of the weights, provided the algebra possesses a diagonalizing subalgebra, a torus. This is how $\mathfrak{gl}(n)$ produces $\mathfrak{gl}(k|n-k)$, or $\mathfrak{e}(6)$, $\mathfrak{e}(7)$, $\mathfrak{e}(8)$ produce their superizations, or $\mathfrak{o}_\Pi(2(n+m))$ produces $\mathfrak{o}_{\Pi\Pi}(2n|2m)$ and $\mathfrak{pe}(2n)$ for $n = m$, or $\mathfrak{h}_\Pi(2n)$ produces $\mathfrak{h}_\Pi(2k|2n-2k)$ and $\mathfrak{le}(n)$, see [LeP, BGL1].

The space V^* (more precisely, $\mathbb{K} \otimes_{\mathbb{F}_2} V^*$, where V^* is considered over $\mathbb{F}_2$) is a torus in the 2-closure of $\mathrm{Kap}_2(2m)$ or $\mathrm{Kap}_{4,A}(2m)$ whereas $u \in V$ is precisely a weight with respect to this torus. That is how we obtain what we call *linear* superizations of the 2-closures of $\mathrm{Kap}_2(2m)$ and $\mathrm{Kap}_{4,A}(2m)$, see below.

The Lie algebras $\mathrm{Kap}_2(2m)$ give us the first examples of how to introduce $\mathbb{Z}/2$-grading *non-linearly*, even in the two non-equivalent ways.

Under any superization — be it linear or not — the even part of the superized subalgebra is a subalgebra of the initial algebra. So there is nothing extraordinary in the fact that the even part of the superization of $\mathrm{Kap}_2(2m) \oplus V^*$ is $\mathrm{Kap}_{4,A}(2m) \oplus V^*$. The whole $\mathrm{Kap}_2(2m)$ can not enter the even part of the superization, since otherwise the odd part would be zero.

5.3.1. **Simple Lie superalgebras $\mathbf{KapS}_{2,A}(2m)$, $\mathbf{KapLS}_2(2m)$ and $\mathbf{KapS}_{4,A}(2m;\varepsilon)$ constructed from Kaplansky algebras $\mathbf{Kap}_2(2m)$ and $\mathbf{Kap}_{4,A}(2m)$.** For basics on Lie superalgebras in characteristic 2, see [LeP, BGL1]. Set (all spaces defined over $\mathbb{F}_2$ are considered over $\mathbb{K}$ by extension of the ground field):

$$\begin{aligned}
&(\mathrm{KapS}_{2,A}(2m))_{\bar 0} := \mathrm{Kap}_{4,A}(2m) \oplus V^*,\\
&(\mathrm{KapS}_{2,A}(2m))_{\bar 1} := \mathrm{Span}(e_u \mid u \in V,\ u \neq 0,\ Q(u) = 0)
\end{aligned} \tag{70}$$

and define the bracket of even elements with any element, and squaring of the odd elements, by means of eqs. (49), (66) and (69).

Same as every known simple Lie algebra has several "hidden supersymmetries" turning it into a Lie superalgebra when several of its appropriately chosen generators are declared odd (as, for example, in [BGL1]), there are several superizations of Kaplansky algebras $\mathrm{Kap}_2(2m)$ and $\mathrm{Kap}_{4,A}(2m)$.

For Kaplansky algebras $\mathrm{Kap}_2(2m)$ and $\mathrm{Kap}_{4,A}(2m)$, there is also another way to introduce superization; let us describe it and isomorphism classes of the Lie superalgebras obtained. Observe that the superization (70) is *non-linear* meaning that parity is not a linear function of u since it is equal to $Q(u) + \bar 1$.

5.3.1a. **On linear superizations of $\mathbf{Kap_2(2\mathit{m})}$ and $\mathbf{Kap_{4,\mathit{A}}(2\mathit{m})}$.** Here we sa "linear" in the sense ythat every e_u is homogenous, and its parity is a linear function of $u \in V$ considered over $\mathbb{F}_2$.

Let the parity be $\varphi \in V^*$, i.e., $p(e_u) = \varphi(u)$. Since the form B is non-degenerate, there is $v \in V$ such that

$$\varphi = B_v, \text{see (67), i.e., } \varphi(u) = B(v, u) \text{ for all } u \in V\text{; we denote this } \varphi \text{ by } \varphi_v. \tag{71}$$

To show that two such superizations induced by distinct non-zero vectors v and v' are isomorphic, it suffices to find a linear map $M : V \longrightarrow V$ such that:

$$\begin{aligned} &1_2) \quad M \text{ preserves } B \text{ for } \mathrm{Kap}_2(2m);\\ &1_4) \quad M \text{ preserves } Q\text{, and therefore preserves } B \text{ as well, for } \mathrm{Kap}_{4,A}(2m);\\ &2) \quad Mv = v'. \end{aligned} \tag{72}$$

Then the induced maps

$$\tilde{M} : e_u \mapsto e_{Mu}; \qquad M^* : \varphi \mapsto \varphi \circ M^{-1} \text{ for any } \varphi \in V^* \tag{73}$$

determine an isomorphism between superizations.

For $\mathrm{Kap}_2(2m)$, such an M exists for any two non-zero v and v' (recall that we consider these vectors over $\mathbb{F}_2$). So, up to an isomorphism, there is one linear superization of $\mathrm{Kap}_2(2m)$ — denote this superization[16] by $\mathrm{KapLS}_2(2m)$. The three Lie superalgebras — $\mathrm{KapLS}_2(2m)$ and $\mathrm{KapS}_{2,A}(2m)$ for $A = 0, 1$ — are non-isomorphic.

For $\mathrm{Kap}_{4,A}(2m)$, such an M exists for two non-zero vectors v and v' if and only if $Q(v) = Q(v')$ (recall that we consider v and v' over $\mathbb{F}_2$). So there are two linear superizations for each $\mathrm{Kap}_{4,A}(2m) \oplus V^*$ with the exception of $\mathrm{Kap}_{4,A}(2)$ where $Q(u) = 1$ for any non-zero u, so there is only one superization $(\mathfrak{oo}'_{II}(1|2))$.[17]

Denote the linear superization of $\mathrm{Kap}_{4,A}(2m) \oplus V^*$ corresponding to a $v \in V$ such that $Q(v) = \varepsilon$ by $\mathrm{KapS}_{4,A}(2m; \varepsilon)$. To describe these Lie superalgebras, recall the definition of the parity φ_v, see (71), so $\varphi_v(u) = B(v, u)$ and consider the following vectors $v = v_{\varepsilon,A} \in V$ assuming that the quadratic forms Q_A are as in eq. (58):

$$\begin{aligned} &v_{0,0} = v_{1,1} = (1, 0, \dots, 0);\\ &v_{1,0} = (1, 0, \dots, 0, 1, 0, \dots, 0) \text{ (the second 1 is in the } (m+1)\text{-st position)};\\ &v_{0,1} = (0, 1, 0, \dots, 0) \text{ for } m > 1 \text{ (so does not exist for } \mathrm{Kap}_{4,1}(2)). \end{aligned} \tag{74}$$

Set

$$\begin{aligned} &\mathrm{KapS}_{4,A}(2m; \varepsilon)_{\bar 0} := \mathrm{Span}(e_u \mid u \neq 0,\ Q_A(u) = 1,\ B(v_{\varepsilon,A}, u) = 0) \oplus V^*,\\ &\mathrm{KapS}_{4,A}(2m; \varepsilon)_{\bar 1} := \mathrm{Span}(e_u \mid u \neq 0,\ Q_A(u) = 1,\ B(v_{\varepsilon,A}, u) = 1). \end{aligned} \tag{75}$$

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[16]It is interesting to find out if $\mathrm{KapS}_2(2m)$ is a deform of a superization of $\mathfrak{h}_\Pi$. This is clearly not so for $\mathfrak{h}_\Pi(2k|2m-2k)$ since their dimensions differ (recall that $\mathrm{KapS}_2(2m)$ contains V^*). But it might be a deform of a larger algebra. **Conjecturally**, it is not.

[17]Actually, the argument with the map (73) does not prove that the two superizations of $\mathrm{Kap}_{4,A}(2m)$ are non-isomorphic, only that there is no isomorphism of the form (73) between them. **Conjecturally**, they are non-isomorphic.

5.3.1b. **There are no non-linear superizations of $\mathrm{Kap}_{4,A}(2m)$ induced by non-linear superizations of $\mathrm{Kap}_2(2m)$.** In $\mathrm{KapS}_{2,A}(2m)$ corresponding to a form Q, take the part corresponding to $\mathrm{Kap}_{4,A}(2m)$ with another form Q'; this is a Lie subsuperalgebra. Can we do so? We can, but fortunately (otherwise the classification would certainly be a nightmare), this superization coincides with a linear one: this subsuperalgebra is singled out by the condition $Q'(u) = 1$ while its even part is singled out by this condition together with an extra condition $Q(u) = 1$ which can be replaced with $Q(u) + Q'(u) = 0$, and since both Q and Q' should yield the same bilinear form B, the quadratic form $Q + Q'$ degenerates into a linear function. So this superization is equivalent to a linear one.

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